9811 Introduction to AEA worksheet

Worked solutions - marked

1. $gf(x) = \left| \dfrac{10}{2x-7} \right| + 3$

(a) $k = \dfrac{7}{2}$ B1

(b) $$gf(x) = \begin{cases} \dfrac{10}{2x-7} + 3 & x \geqslant \dfrac{7}{2} \\ \dfrac{10}{7-2x} + 3 & x < \dfrac{7}{2} \end{cases}$$

In the region R, $x < \dfrac{7}{2}$

$$\therefore \quad \text{Area}(R) = \int_{-\frac{9}{2}}^{-1} \left(\frac{10}{7-2x} + 3 \right) dx$$

$$= \left[\frac{1}{-2} \cdot 10 \ln(7-2x) + 3x \right]_{-\frac{9}{2}}^{-1}$$

M1 A1

$$= \left[-5 \ln(7-2x) + 3x \right]_{-\frac{9}{2}}^{-1}$$

$$= -5\ln 9 - 3 + 5\ln 16 + \frac{27}{2}$$ M1

$$= 5\ln\frac{16}{9} + \frac{21}{2}$$ as required. A1

$\therefore\ a = 5,\ b = 16,\ c = 9,\ d = \frac{21}{2}$

2. $y = \sin x \cdot e^{\sin x} + a$

M1

(a) Note $|\sin x| \leq 1$, $e^x > 0$

Minimum value of y occurs when $\sin x = -1$

$y = -1 \cdot e^{-1} + a = 0 \Rightarrow a = \frac{1}{e}$ A1

Maximum value of y occurs when $\sin x = 1$

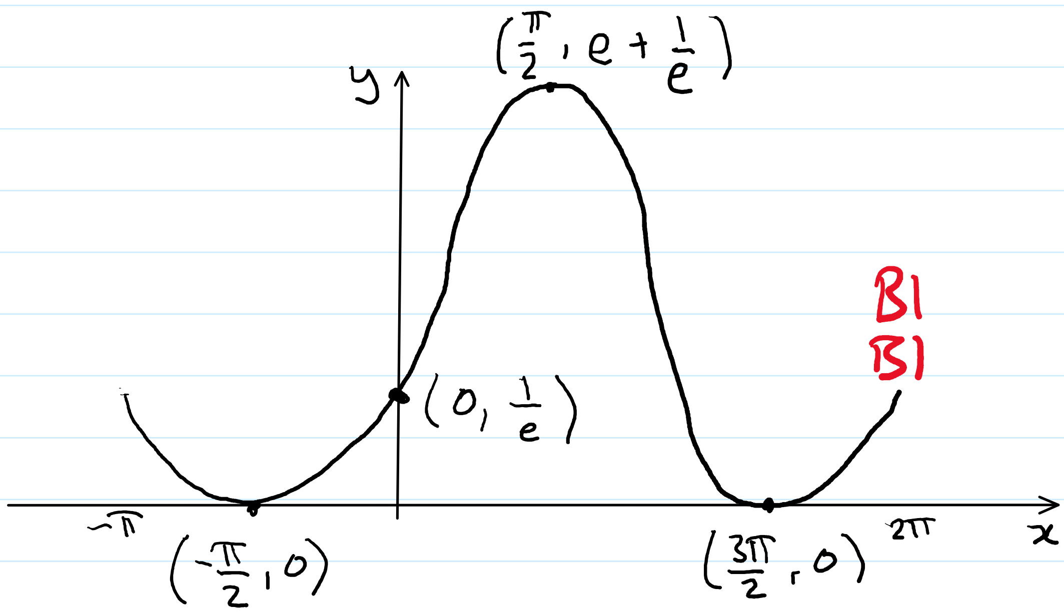
$y = 1 \cdot e^{1} + a = b \Rightarrow b = e + \frac{1}{e}$ A1

(b) $y = \sin x \cdot e^{\sin x} + \frac{1}{e}$

In $-\pi \leq x \leq 2\pi$, $\sin x = -1$ when $x = -\frac{\pi}{2}, \frac{3\pi}{2}$

$\sin x = 1$ when $x = \frac{\pi}{2}$

When $x = 0$, $y = \frac{1}{e}$



3.

(a) From the formula book: for $|-x|<1$ i.e. $|x|<1$,

$$(1-x)^{-4} = 1 + (-4)(-x) + \dots + \frac{(-4)(-5)(-6)\dots(-4-r+1)}{1\times 2\times 3\times\dots\times r}(-x)^r + \dots$$ B1

$\therefore$ x^r term is $\dfrac{(-4)(-5)(-6)\dots(-r-3)(-1)^r x^r}{r!}$

$$= \frac{4\times 5\times 6\times\dots\times(r+3)\,x^r}{r!}$$ M1

$$= \frac{(r+3)!}{3!\,r!}x^r$$

$$= \frac{(r+1)(r+2)(r+3)\,x^r}{6}$$

$\therefore$ The coefficient of x^r is $\dfrac{(r+1)(r+2)(r+3)}{6}$ as required.

A1

(b) The coefficient of x^r in $(3+2x-5x^2)(1-x)^{-4}$ is

$$3\times\frac{(r+1)(r+2)(r+3)}{6} + 2\times\frac{r(r+1)(r+2)}{6} - 5\times\frac{(r-1)r(r+1)}{6}$$ B1

$$= \frac{r+1}{6}\left[3(r+2)(r+3) + 2r(r+2) - 5(r-1)r\right]$$

$$= \frac{r+1}{6}\left(3r^2+15r+18 + 2r^2+4r - 5r^2+5r\right)$$ M1

$$= \frac{r+1}{6}\left(24r+18\right)$$

$= (r+1)(4r+3)$ as required, with $A = 4$, $B = 3$

A1

c) $S = \sum_{r=0}^{\infty} \left((r+1)(4r+3) x^r \right)$

$= 1 \times 3 + 2 \times 7x + 3 \times 11x^2 + 4 \times 15x^3 + 5 \times 19x^4 + \ldots$

$= 3 + 14x + 33x^2 + 60x^3 + 95x^4 + \ldots$ M1

$= 3 - 7 + \frac{33}{4} - \frac{15}{2} + \frac{95}{16} - \ldots$

$\Rightarrow \quad x = -\frac{1}{2}$

$\therefore \quad S = \left(3 + 2(-\tfrac{1}{2}) - 5(-\tfrac{1}{2})^2\right)\left(1 - (-\tfrac{1}{2})\right)^{-4}$ dM1

$= \frac{3}{4} \times \left(\frac{3}{2}\right)^{-4} = \frac{3}{4} \times \left(\frac{2}{3}\right)^{4} = \frac{4}{27}$

A1

(d) $3 + 14x + 33x^2 + 60x^3 + 95x^4 + \ldots$

$= 3 - 28 + 132 - 480 + 1520 + \ldots$

would require $x = -2$, but the series expansion is only valid for $|x| < 1$ and would diverge for $x = -2$. As a result, the student would be unsuccessful using this approach.

B1

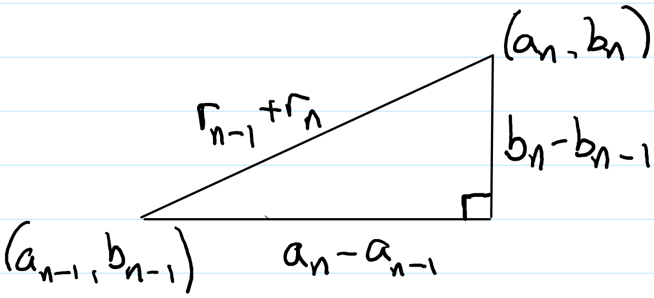
4.

(a) $n = 1 \Rightarrow r = 2 \Rightarrow$ centre is $(2,2)$ so that C_1 touches both the x-axis and y-axis.

$\therefore$ Equation of C_1 is $(x-2)^2 + (y-2)^2 = 4$

M1A1

(b) Adjacent centres would be (a_{n-1}, b_{n-1}) and (a_n, b_n).



By Pythagoras:

$$(a_n - a_{n-1})^2 + (b_n - b_{n-1})^2 = (r_n + r_{n-1})^2 \quad \text{M1}$$

and note $b_n = r_n$

$$\therefore (a_n - a_{n-1})^2 + b_n^2 - 2b_n b_{n-1} + b_{n-1}^2 = b_n^2 + 2b_n b_{n-1} + b_{n-1}^2$$

$$\Rightarrow (a_n - a_{n-1})^2 = 4b_n b_{n-1} \quad \text{dM1} \quad a_n > a_{n-1}$$

$$\therefore a_n - a_{n-1} = 2\sqrt{b_n b_{n-1}} \quad \text{as required.}$$

A1

(c) Note $r_n = 2^n$ and $b_n = r_n$

From (b) $a_n = a_{n-1} + 2\sqrt{2^n . 2^{n-1}}$ M1

$$= a_{n-1} + 2\sqrt{2}\sqrt{2^{n-1}.2^{n-1}}$$

$$= a_{n-1} + 2\sqrt{2} . 2^{n-1}$$

$= a_{n-1} + 2^n\sqrt{2}$ as required with $f(n) = 2^n$

A1

(d) The gradient between centres of C_{n-1} and C_n is

$$m = \frac{b_n - b_{n-1}}{a_n - a_{n-1}}$$

$$= \frac{2^n - 2^{n-1}}{2^n\sqrt{2}}$$ M1 using $b_n = 2^n$ and the result from part (c)

$$= \frac{2^{n-1}(2-1)}{2^n\sqrt{2}} = \frac{1}{2\sqrt{2}}$$ A1

which is independent of n.

As a result, the gradient between any two adjacent centres is the same.

Each line segment joining successive centres has a shared point and since the gradients are the same the centres must lie on a straight line. A1 (S+)

(e) C_1 has centre $(2, 2)$

The line, l, passing through centres has gradient $\frac{1}{2\sqrt{2}}$

$$\therefore l: y - 2 = \frac{1}{2\sqrt{2}}(x-2)$$ B1

and since C_n lies on l we have

$$2^n - 2 = \frac{1}{2\sqrt{2}}(a_n - 2)$$ M1

$$\Rightarrow 2^{n+1}\sqrt{2} - 4\sqrt{2} = a_n - 2$$

$$a_n = 2 + 4(2^{n-1}) - 4\sqrt{2}$$

$$a_n = 2 + 4\sqrt{2}(2^{n-1} - 1)$$ as required.

A1 S1

5.

(a) Let T be the time at which the particle is at (x, y)

$(\rightarrow)$ $s = x$, $u = U\cos\alpha$, $t = T$

$(\uparrow)$ $s = y$, $u = U\sin\alpha$, $v =$, $a = -g$, $t = T$

Note: $s_0 = H$

$\therefore \quad x = UT\cos\alpha$ B1

$\Rightarrow \quad T = \dfrac{x}{U\cos\alpha}$

$$s = s_0 + ut + \frac{1}{2}at^2$$

$$y = H + UT\sin\alpha + \frac{1}{2}(-g)T^2$$

B1

$$y = H + x\frac{\cancel{U}\sin\alpha}{\cancel{U}\cos\alpha} - \frac{g}{2}\left(\frac{x}{U\cos\alpha}\right)^2$$ M1

$$y = H + x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha}$$

$$y = H + x\tan\alpha - \frac{gx^2\sec^2\alpha}{2U^2}$$ dM1

$$y = H + x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha)$$ as required.

A1

(b) We require $y = 0$ to find the distance travelled

Further, $x_{max} = R$ when $\alpha = \beta$ requires $\dfrac{dx}{d\alpha} = 0$

$$y = 0 \Rightarrow \quad 0 = H + x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha)$$

Differentiating implicitly with respect to α:

$$0 = \frac{dx}{d\alpha}\tan\alpha + x\sec^2\alpha - \frac{\cancel{2}gx}{\cancel{2}U^2}(1+\tan^2\alpha)\frac{dx}{d\alpha} - \frac{gx^2}{\cancel{2}U^2}(\cancel{2}\tan\alpha\sec^2\alpha)$$

M1A1

Since $\dfrac{dx}{d\alpha} = 0$ when $x = R$, $\alpha = \beta$, this becomes

$$0 = R\sec^2\beta - \frac{gR^2}{U^2}\tan\beta\sec^2\beta$$ dM1 (St)

$$0 = R\sec^2\beta\left(1 - \frac{gR}{U^2}\tan\beta\right)$$

so either $R = 0$, $\sec\beta = 0$ or $gR\tan\beta = U^2$

Hence $R = \dfrac{U^2\cot\beta}{g}$ as required. ddM1 A1

(c) Using the result from part (a):

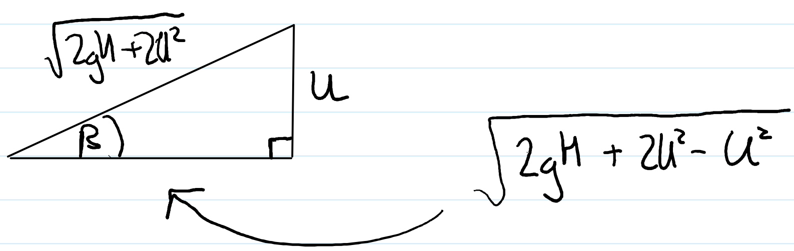
$$0 = H + \frac{\cancel{R}U^2}{g\cancel{R}} - \frac{gR^2}{2U^2}(1+\tan^2\beta)$$ M1

$$0 = H + \frac{U^2}{g} - \frac{g\left(\frac{U^2\cot\beta}{g}\right)^2\sec^2\beta}{2U^2}$$

$$0 = H + \frac{U^2}{g} - \frac{U^2\cot^2\beta\sec^2\beta}{2g}$$

$$0 = H + \frac{U^2}{g} - \frac{U^2\operatorname{cosec}^2\beta}{2g}$$ dM1 A1 (s+)

$$\operatorname{cosec}^2\beta = \frac{2gH + 2U^2}{U^2} \quad \text{i.e. } \sin\beta = \frac{U}{\sqrt{2gH+2U^2}}$$



$$\therefore \tan\beta = \frac{U}{\sqrt{2gH+U^2}}$$ ddM1

and $\beta = \arctan\left(\dfrac{U}{\sqrt{2gH+U^2}}\right)$ as required. A1 S1